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Shrinking Projection Methods for Variational Inequalities in Banach Spaces*

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Abstract: In uniformly smooth and uniformly convex Banach spaces, a shrinking projection algorithm is proposed for finding an element of the solution set of variational inequalities, and a strong convergence theorem is proved by using the generalized projection operator, K-K property and other analysis techniques under the conditions of compact mappings weakening continuous mappings. The results of this paper improve and extend recent some relevant results. The proposed algorithm has important applications.

Keywords: shrinking projection method; variational inequalities; K-K property

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1 Introduction and preliminaries

Throughout this paper, we assume that X is a Banach space and X^* its dual space. We use $\langle\cdot,\cdot\rangle$ to denote the duality pairing between X^* and X. Let C be a subset of X and $T: C \to X^*$ be a mapping. We consider the following variational inequality problem: find $x \in C$ such that

$$\langle Tx, y-x\rangle \geq 0 \tag{1}$$

for all $y \in C$. A point $x_0 \in C$ is called a solution of the variational inequality (1) if $\langle Tx_0, y - x_0\rangle \geq 0$ for all $y \in C$. The solution set of the variational inequality (1) is denoted by $VI(C,T)$.

The variational inequality (1) has been extensively investigated[1-6]. Let X, Y be Banach spaces. A mapping $T: D(T) \subset X \to Y$ is said to be compact if it is continuous and maps the bounded subsets of $D(T)$ onto the relatively compact subsets of Y, where $D(T)$ is the domain of T. Recently, Fan[7] proved the following theorem.

Theorem 1 Let X be a uniformly convex and uniformly smooth Banach space and let C be a closed convex subset of X. Suppose that there exists a positive number β such that $\langle Tx, J^{-1}(Jx-\beta Tx)\rangle \geq 0$ for all $x \in C$ and $J-\beta T: C \to X^*$ is compact. If $\langle Tx, y\rangle \leq 0$ for all $x \in C$ and $y \in VI(C,T)$, then the variational inequality (1) has a solution $x^* \in C$. The sequence $\{x_n\}$ defined by the following iterative scheme

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\pi_C(Jx_n - \beta Tx_n), \quad n = 1, 2, \cdots,$$

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where $\{\alpha_n\}$ satisfies $0 < a \le \alpha_n \le b < 1$ for all $n \in \mathbb{N}$ and for some positive numbers a, $b \in (0,1)$ such that $a < b$, $\mathbb{N}$ is the set of positive integers. Then $\{x_n\}$ converges strongly $x^* \in C$.

Question Can the compact mapping $J - \beta T$ in Theorem 1 be weaken to the continuous mapping $J - \beta T$?

The purpose of this paper is to solve the above Question by introducing a new and simple shrinking projection method. The results of this paper improve and extend the corresponding results of Li[6], Fan[7] and others.

We denote by J the normalized duality mapping from X to 2^{X^*} defined by

$$Jx = \{f \in X^* : \langle f, x\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall\, x \in X.$$

Recall that a Banach space X has the K-K property if for any sequence $\{x_n\} \subset X$ that converges weakly to x where also $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ ([8]). It is known that every uniformly convex Banach space has the K-K property.

Let R be the set of real numbers. The functional $V : X^* \times X \to R$ is defined by

$$V(\phi, x) = \|\phi\|^2 - 2\langle \phi, x\rangle + \|x\|^2,$$

where $\phi \in X^*$ and $x \in X$ ([2]). The functional $V_2 : X \times X \to R$ is defined by $V_2(x,y) = V(Jx, y)$, for all x, $y \in X$. Let X be a reflexive, strictly convex and smooth Banach space. Then the generalized projection operator $\pi_C : X^* \to C$ is continuous ([9]). The generalized projection operator π_C and the functional V have the following properties,

(i) $V : X^* \times X \to R$ is continuous;

(ii) $V(\phi, x) = 0$ if and only if $\phi = Jx$;

(iii) $V(J\pi_C\phi, x) \le V(\phi, x)$ for all $\phi \in X^*$ and $x \in X$;

(iv) $\pi_C(Jx) = x$ for all $x \in C$;

(v) Let X be smooth. For any given $\phi \in X^*$ and $x \in C$, $x \in \pi_C\phi$ if and only if $\langle \phi - Jx,\, x - y\rangle \ge 0$ for all $y \in C$;

(vi) The operator $\pi_C : X^* \to C$ is single-valued if and only if X is strictly convex;

(vii) Let X be smooth. If $x \in \pi_C\phi$, then $V(Jx, y) \le V(\phi, y) - V(\phi, x)$, for all $\phi \in X^*$, $y \in C$ ([1,9]).

Remark 1 If X is a reflexive, strictly convex and smooth Banach space, then for x, $y \in X$, $V_2(x,y) = 0$, i.e., $V(Jx, y) = 0$ if and only if $x = y$.

Lemma 1[7] Let C be a nonempty closed and convex subset of a reflexive, strictly convex and smooth Banach space X and let $\phi \in X^*$. Then there exists a unique element in C, denoted by $\pi_C\phi$, such that $V(\phi, \pi_C\phi) = \inf_{y\in C} V(\phi, y)$.

Lemma 2[1] Let X be a reflexive, strictly convex and smooth Banach space with dual space X^*. Let T be an arbitrary operator from X to X^* and let α be an arbitrary fixed positive number. Then the point $x \in C \subset X$ is a solution to the variational inequality (1) if and only if x is a solution of the operator equation in X, $x = \pi_C(Jx - \alpha Tx)$.

Lemma 3[5] Let X be a uniformly convex Banach space and $B_r(0)$ be a closed ball of X. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$V_2(\alpha x_1 + (1-\alpha)x_2, y) \le \alpha V_2(x_1, y) + (1-\alpha)V_2(x_2, y) - \alpha(1-\alpha)g(\|x_1 - x_2\|),$$

for all x_1, x_2, $y \in B_r(0)$ and $\alpha \in [0,1]$.

Lemma 4[10] Let X be a uniformly convex and smooth Banach space and $\{y_n\}$ and $\{z_n\}$ be two sequences of X. If $V_2(z_n, y_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $z_n - y_n \to 0$.

Lemma 5[4] Let X be a uniformly convex and uniformly smooth Banach space. Then the following inequality holds

$$\|\phi + f\|^2 \le \|\phi\|^2 + 2\langle \phi,\, J^{-1}(\phi + f)\rangle, \quad \forall\, \phi, f \in X^*.$$

Lemma 6 Let X be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed and convex subset of X. If there exists a positive number β such that

$$\langle Tx,\, J^{-1}(Jx - \beta Tx)\rangle \ge 0, \quad \forall\, x \in C, \tag{2}$$

$$\langle Tx,\, y\rangle \le 0, \quad \forall\, x \in C, \quad y \in VI(C,T). \tag{3}$$

Then $VI(C,T)$ is closed and convex.

Proof From the definition of V_2, the property (iii) of V, (2), (3) and Lemma 4, we can obtain that $VI(C,T)$ is closed and convex.

2 Main results

Theorem 2 Let X be a uniformly convex and uniformly smooth Banach space. Let C be a nonempty closed convex subset of X. Assume that T is an operator of C into X^* which satisfies conditions (2) and (3). Define a sequence $\{x_n\}$ by the following algorithm

$$\begin{cases} x_0 \in X, \quad C_1 = C, \\ y_n = (1-\alpha_n)x_n + \alpha_n \pi_C(Jx_n - \beta Tx_n), \\ C_{n+1} = \{z \in C_n : V_2(y_n, z) \le V_2(x_n, z)\}, \\ x_{n+1} = \pi_{C_{n+1}} Jx_0, \end{cases} \tag{4}$$

where $\{\alpha_n\} \subset (0,1]$ satisfies $\liminf\limits_{n\to\infty} \alpha_n > 0$. If $J - \beta T : C \to X^*$ is continuous, then $\{x_n\}$ converges strongly to $\pi_{VI(C,T)}Jx_0$.

Proof 1) Show that $\pi_{VI(C,T)}Jx_0$ is well defined for every $x_0 \in X$.

From [7], we know that $VI(C,T) \ne \emptyset$. By Lemma 6 and Lemma 1, we have that $\pi_{VI(C,T)}Jx_0$ is well defined.

2) Show that C_n is closed and convex for all $n \in \mathbb{N}$.

This follows from the construction of C_n. We omit the details.

3) Show that $VI(C,T) \subset C_n$ for all $n \in \mathbb{N}$.

It is obvious that $VI(C,T) \subset C = C_1$. Suppose that $VI(C,T) \subset C_n$ for some $n \in \mathbb{N}$. For any $p \in VI(C,T) \subset C_n$, from Lemma 3, we have

$$\begin{aligned} V_2(y_n, p) &= V_2\big((1-\alpha_n)x_n + \alpha_n \pi_C(Jx_n - \beta Tx_n),\, p\big) \\ &\le (1-\alpha_n)V_2(x_n, p) + \alpha_n V_2\big(\pi_C(Jx_n - \beta Tx_n),\, p\big) \\ &\quad -\alpha_n(1-\alpha_n)g\big(\|\pi_C(Jx_n - \beta Tx_n) - x_n\|\big). \end{aligned} \tag{5}$$

From the definition of V_2, the property (iii) of V, (2), (3) and Lemma 5, we have

$$\begin{aligned} V_2\big(\pi_C(Jx_n - \beta Tx_n), p\big) &= V\big(J\pi_C(Jx_n - \beta Tx_n), p\big) \\ &\le \|Jx_n - \beta Tx_n\|^2 - 2\langle Jx_n - \beta Tx_n, p\rangle + \|p\|^2 \\ &\le \|Jx_n\|^2 - 2\langle Jx_n, p\rangle + \|p\|^2 = V_2(x_n, p). \end{aligned} \tag{6}$$

From (5) and (6), we obtain that $V_2(y_n, p) \le V_2(x_n, p)$, which implies that $p \in C_{n+1}$ and hence $VI(C,T) \subset C_{n+1}$. Therefore, $VI(C,T) \subset C_n$ for all $n \in \mathbb{N}$.

4) Show that $\lim\limits_{n\to\infty} V(Jx_0, x_n)$ exists.

In view of (4), we have $x_n = \pi_{C_n} Jx_0$. Since $C_{n+1} \subset C_n$ and $x_{n+1} \in C_{n+1}$ for all $n \in \mathbb{N}$, we have $V(Jx_0, x_n) \le V(Jx_0, x_{n+1})$. On the other hand, we have from 3) that $V(Jx_0, x_n) \le V(Jx_0, p)$ for all $p \in VI(C,T)$. It follows that $\lim\limits_{n\to\infty} V(Jx_0, x_n)$ exists.

5) Show that $x_n \to p_0 \in C$ as $n \to \infty$.

From 4), we have $\{x_n\}$ is bounded. Note that X is reflexive, without loss of generality, we can assume that $x_n \to p_0$ weakly as $n \to \infty$ (passing to a subsequence if necessary). It is easy to see that $p_0 \in C_n$ for all $n \in \mathbb{N}$. Noticing that $V(Jx_0, x_n) \le V(Jx_0, x_{n+1}) \le V(Jx_0, p_0)$, by using the definition of V and weakly lower semi-continuity of $\|\cdot\|^2$, we obtain that

$$V(Jx_0, p_0) \le \liminf_{n\to\infty} V(Jx_0, x_n) \le \limsup_{n\to\infty} V(Jx_0, x_n) \le V(Jx_0, p_0).$$

It follows that $V(Jx_0, x_n) \to V(Jx_0, p_0)$ as $n \to \infty$. Hence $\|x_n\| \to \|p_0\|$ as $n \to \infty$. Since X has the K-K property, we have $x_n \to p_0$ as $n \to \infty$.

6) Show that $p_0 \in VI(C,T)$.

Noticing that $x_n = \pi_{C_n} Jx_0$ and $x_{n+1} \in C_{n+1} \subset C_n$, in view of the property (vii), we have

$$V(Jx_n, x_{n+1}) \le V(Jx_0, x_{n+1}) - V(Jx_0, x_n), \quad \forall\, n \in \mathbb{N}.$$

From 4), we have

$$\lim_{n\to\infty} V(Jx_n, x_{n+1}) = \lim_{n\to\infty} V_2(x_n, x_{n+1}) = 0.$$

Since $V_2(y_n, x_{n+1}) \le V_2(x_n, x_{n+1})$, we have $\lim\limits_{n\to\infty} V_2(y_n, x_{n+1}) = 0$. By using Lemma 4, we obtain that

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{7}$$

On the other hand, from (4), we have

$$\begin{aligned} \|x_{n+1} - y_n\| &= \big\|(1-\alpha_n)(x_{n+1} - x_n) + \alpha_n\big(x_{n+1} - \pi_C(Jx_n - \beta Tx_n)\big)\big\| \\ &\ge \alpha_n\big\|x_{n+1} - \pi_C(Jx_n - \beta Tx_n)\big\| - (1-\alpha_n)\|x_{n+1} - x_n\|. \end{aligned}$$

From the condition $\liminf\limits_{n\to\infty} \alpha_n > 0$ and (7), we obtain that $\|x_{n+1} - \pi_C(Jx_n - \beta Tx_n)\| \to 0$ as $n \to \infty$. Since $x_n \to p_0$ from 5), we have $\lim\limits_{n\to\infty} \pi_C(Jx_n - \beta Tx_n) = p_0$. Noticing that π_C and $J - \beta T$ are continuous, we have

$$\lim_{n\to\infty} \pi_C(Jx_n - \beta Tx_n) = \pi_C(Jp_0 - \beta Tp_0).$$

It follows that $\pi_C(Jp_0 - \beta Tp_0) = p_0$. By Lemma 2, we have $p_0 \in VI(C,T)$.

7) Show that $p_0 = \pi_{VI(C,T)} Jx_0$.

From $x_n = \pi_{C_n} Jx_0$, one sees $\langle Jx_0 - Jx_n, x_n - y\rangle \geq 0$, for all $y \in C_n$. From 3), we have

$$\langle Jx_0 - Jx_n, x_n - w\rangle \geq 0, \quad \forall\, w \in VI(C,T).$$

Since $J : X \to X^*$ is demi-continuous, we have $\langle Jx_0 - Jp_0, p_0 - w\rangle \geq 0$, for all $w \in VI(C,T)$. It follows from the property (v) of π_C that $p_0 = \pi_{VI(C,T)} Jx_0$.

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Banach空间中关于变分不等式的收缩投影方法

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摘　要: 在一致光滑的一致凸的Banach空间中，设计了一种收缩投影算法用以逼近变分不等式的解，并在紧算子减弱为连续算子的条件下，利用广义投影算子和K-K性质等技巧证明了该算法的强收敛性. 所得结果是近期相关结果的改进与推广，其算法有重要应用.

关键词: 收缩投影算法；变分不等式；K-K性质